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MAT-19961 Calcul matriciel en génie

Solutions - Devoir 12

1.

$$\hat{\mathbf{y}} = \text{proj}_W\mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\right)\mathbf{u}_3$$

$$= \left(\frac{4+3+0-1}{1+1+0+1}\right)\mathbf{u}_1 + \left(\frac{-4+9+3+6}{1+9+1+4}\right)\mathbf{u}_2 + \left(\frac{-4+0+3-1}{1+0+1+1}\right)\mathbf{u}_3$$

$$= 2\mathbf{u}_1 + \left(\frac{2}{3}\right)\mathbf{u}_2 + \left(-\frac{2}{3}\right)\mathbf{u}_3$$

$$= 2\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \left(\frac{2}{3}\right)\begin{bmatrix}-1\\3\\1\\-2\end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix}-1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\4\\0\\0\end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}4\\3\\3\\-1\end{bmatrix} - \begin{bmatrix}2\\4\\0\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\3\\-1\end{bmatrix}$$

2. Soit $W = [\mathbf{v}_1\ \mathbf{v}_2]$. La meilleure approximation de $\mathbf{z}$ dans W est $\text{proj}_W\mathbf{z}$.

$$\text{proj}_W\mathbf{z} = \left(\frac{\mathbf{z}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{z}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2$$

$$= \left(\frac{6+7-6+3}{4+1+9+1}\right)\mathbf{v}_1 + \left(\frac{3-7+2-3}{1+1+0+1}\right)\mathbf{v}_2$$

$$= \left(\frac{2}{3}\right)\mathbf{v}_1 + \left(-\frac{5}{3}\right)\mathbf{v}_2 = \left(\frac{2}{3}\right)\begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} + \left(-\frac{5}{3}\right)\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}$$

3. a)

$$U^TU = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$UU^T = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

b)

$$\begin{aligned} \text{proj}_W\mathbf{y} &= \left(\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 \\ &= \left(\frac{\frac{8}{3}+\frac{8}{3}+\frac{2}{3}}{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}}\right)\mathbf{u}_1 + \left(\frac{-\frac{8}{3}+\frac{16}{3}+\frac{1}{3}}{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}\right)\mathbf{u}_2 \\ &= 6\mathbf{u}_1 + 3\mathbf{u}_2 = 6\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} + 3\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \end{aligned}$$

$$UU^T\mathbf{y} = \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

4.

```
>>U

U =

   -0.6000   -0.3000    0.6000    0.1000
   -0.1000    0.2000    0.1000   -0.6000
    0.3000    0.6000    0.3000   -0.2000
    0.6000   -0.3000    0.6000   -0.1000
    0.2000   -0.1000    0.2000    0.3000
   -0.3000    0.6000    0.3000    0.2000
   -0.2000   -0.1000    0.2000   -0.3000
    0.1000    0.2000    0.1000    0.6000

>>y'

ans =

     1     1     1     1     1     1     1     1

>>(U*U'*y)'

ans =

 Columns 1 through 7

    1.2000    0.4000    1.2000    1.2000    0.4000    1.2000    0.4000

 Column 8

    0.4000
```

Il ne faut pas oublier de normaliser les colonnes de la matrice du problème 6.2.33. Comme la matrice U a des colonnes orthonormales, on peut utiliser le théorème. Le point le plus proche de $\mathbf{y}$ dans Col U est la projection orthogonale de $\mathbf{y}$ dans Col U.

5.

$R = Q^T A$ et on vérifie en faisant $QR = A$.

```
>>format rat
>>A

A =

      5              9
      1              7
```

```
      -3             -5
       1              5

>>Q

Q =

     5/6          -1/6
     1/6           5/6
    -1/2           1/6
     1/6           1/2

>>format short
>>R=Q'*A

R =

    6.0000   12.0000
    0.0000    6.0000

>>Q*R-A

ans =

 1.0e-015 *

    0.8882         0
         0    0.8882
   -0.4441         0
         0         0
```

6. Il nous faut trouver Q et R.

Calcul de Q:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \left(\frac{2-1-4-4+2}{5}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \left(\frac{5+4+3+7+1}{5}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{15+0-9-21+3}{9+0+9+9+9}\right)\begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{1}{3}\right)\begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

En normalisant les vecteurs $\mathbf{v}_1$, $\mathbf{v}_2$ et $\mathbf{v}_3$, on obtient la matrice Q.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{5}} & 0 & 0 \\ -\frac{1}{\sqrt{5}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{5}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

7.

```
A =

   -10    13     7   -11
     2     1    -5     3
    -6     3    13    -3
    16   -16    -2     5
```

```
     2     1    -5    -7

>>v1=A(:,1)/norm(A(:,1))

v1 =

   -0.5000
    0.1000
   -0.3000
    0.8000
    0.1000

>>v2=A(:,2)-(v1*v1')*A(:,2)

v2 =

    3.0000
    3.0000
   -3.0000
    0.0000
    3.0000

>>v2=v2/norm(v2)

v2 =

    0.5000
    0.5000
   -0.5000
    0.0000
    0.5000

>>Q=[v1 v2];
>>v3=A(:,3)-(Q*Q')*A(:,3)

v3 =

    6.0000
    0.0000
    6.0000
    6.0000
    0.0000

>>v3=v3/norm(v3)

v3 =
```

```
     0.5774
     0.0000
     0.5774
     0.5774
     0.0000

>>Q=[Q v3];
>>v4=A(:,4)-(Q*Q')*A(:,4)

v4 =

   -0.0000
    5.0000
         0
    0.0000
   -5.0000

>>v4=v4/norm(v4)

v4 =

   -0.0000
    0.7071
         0
    0.0000
   -0.7071

>>Q=[Q v4]

Q =

   -0.5000    0.5000    0.5774   -0.0000
    0.1000    0.5000    0.0000    0.7071
   -0.3000   -0.5000    0.5774         0
    0.8000    0.0000    0.5774    0.0000
    0.1000    0.5000    0.0000   -0.7071

>>R=Q'*A

R =

   20.0000 -20.0000 -10.0000   10.0000
    0.0000    6.0000   -8.0000   -6.0000
    0.0000   -0.0000   10.3923   -5.1962
    0.0000   -0.0000   -0.0000    7.0711
```

```
>>A-Q*R

ans =

 1.0e-013 *

   -0.0888    0.0888    0.0444   -0.0178
   -0.1243    0.1266    0.0266   -0.0488
         0   -0.0044    0.0355   -0.0089
   -0.0355         0    0.0444         0
    0.0377   -0.0466   -0.0444    0.0355
```