

## MAT-19961 Calcul matriciel en génie

### Solutions - Devoir 11

#### 1. 6.3.8

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = -1 + 3 - 2 = 0, \mathbf{u}_1 \perp \mathbf{u}_2$$

$$W = \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2 \}$$

$$\text{proj}_W \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = 3, \mathbf{u}_2 \cdot \mathbf{u}_2 = 14$$

$$\mathbf{y} \cdot \mathbf{u}_1 = 6, \mathbf{y} \cdot \mathbf{u}_2 = 7$$

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y} = \left(\frac{6}{3}\right)\mathbf{u}_1 + \left(\frac{7}{14}\right)\mathbf{u}_2 = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

$$\text{Donc, } \hat{\mathbf{y}} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

## 2. 6.3.15

$$\mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$$

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 9 - 10 + 1 = 0, \mathbf{u}_1 \perp \mathbf{u}_2$$

Trouvons  $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ , où  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ .

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = 9 + 25 + 1 = 35, \mathbf{u}_2 \cdot \mathbf{u}_2 = 9 + 4 + 1 = 14$$

$$\mathbf{y} \cdot \mathbf{u}_1 = -15 + 45 + 5 = 35$$

$$\mathbf{y} \cdot \mathbf{u}_2 = -15 - 18 + 5 = -28$$

$$\hat{\mathbf{y}} = \left(\frac{35}{35}\right) \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + \left(\frac{-28}{14}\right) \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

La distance entre  $\mathbf{y}$  et  $W$  est donnée par

$$\begin{aligned} \|\mathbf{y} - \hat{\mathbf{y}}\| &= [(5 - 3)^2 + (-9 + 9)^2 + (5 + 1)^2]^{1/2} \\ &= (4 + 0 + 36)^{1/2} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

## 3. 6.3.22

a) Vrai. Voir la preuve du théorème 8, page 390 du livre de Lay.

b) Vrai. Voir la section au bas de la page 391 du livre de Lay. ("A Geometric Interpretation of the Ortho-

gonal Projection").

- c) Vrai. La décomposition orthogonale est unique.
- d) Faux. Selon le théorème 9, page 392 du livre de Lay, la meilleure approximation est  $\text{proj}_W \mathbf{y}$ .
- e) Faux, sauf si  $n = p$ , parce que  $UU^T \mathbf{x}$  est la seule projection orthogonale de  $\mathbf{x}$  sur l'espace des colonnes de  $U$ . Voir le paragraphe qui suit le théorème 10, page 394 du livre de Lay.

#### 4.

```
>>A
```

```
A =
```

```
  -6   -3    6    1
  -1    2    1   -6
   3    6    3   -2
   6   -3    6   -1
   2   -1    2    3
  -3    6    3    2
  -2   -1    2   -3
   1    2    1    6
```

```
>>b'
```

```
ans =
```

```
  1   -1    1   -1    1   -1    1   -1
```

```
>>bc=A*A'*b;
```

```
>>bc'
```

```
ans =
```

```
  64    8   -44   -4   -8   -16   28   -28
```

```
>>norm(b-bc)
```

```
ans =
```

```
 88.5889
```

5.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3]$$

$$\mathbf{v}_1 = \mathbf{a}_1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \left( \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left( \frac{-40}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \left( \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left( \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \left( \frac{30}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \left( \frac{-10}{20} \right) \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Normalisation:

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}$$

$$Q = [\mathbf{v}'_1 \mathbf{v}'_2 \mathbf{v}'_3]$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 3/2 & 1/2 & -3/2 \\ 1/2 & 3/2 & 1/2 \\ -1/2 & 3/2 & 1/2 \\ 3/2 & -1/2 & 3/2 \end{bmatrix}$$

$$R = Q^T A = \frac{1}{\sqrt{5}} \begin{bmatrix} 10 & -20 & 15 \\ 0 & 10 & -5 \\ 0 & 0 & 10 \end{bmatrix}$$

### 6. 6.4.18

- a) Faux. Les trois vecteurs orthogonaux doivent être non nuls pour être une base d'un sous-espace tridimensionnel.
- b) Vrai. Si  $\mathbf{x} \notin W$ , alors  $\mathbf{x} \neq \text{proj}_W \mathbf{x}$  car  $\text{proj}_W \mathbf{x} \in W$ .
- c) Vrai. Voir le théorème 12, page 400 du livre de Lay.

### 7. 6.4.24

On peut utiliser la fonction Matlab suivante.

```
function [Q, R]=gramm(A)

Q=A(:,1)/norm(A(:,1));
for j=2:size(A,2)
    v=A(:,j)-Q*(Q'*A(:,j));
    Q(:,j)=v/norm(v);
end
R=Q'*A;
```

*Exemple d'utilisation avec la matrice du numéro 6.4.22.*

A =

```
-10    13     7    -11
  2     1    -5     3
 -6     3    13    -3
 16   -16    -2     5
  2     1    -5    -7
```

```
>>[Q R]=gramm(A)
```

Q =

```
-0.5000    0.5000    0.5774    0.0000
 0.1000    0.5000    0.0000    0.7071
-0.3000   -0.5000    0.5774    0.0000
```

```
0.8000      0      0.5774      0.0000
0.1000      0.5000      0.0000      -0.7071
```

R =

```
20.0000  -20.0000  -10.0000  10.0000
 0.0000   6.0000  -8.0000  -6.0000
-0.0000   0.0000  10.3923  -5.1962
-0.0000   0.0000   0.0000   7.0711
```

>>Q'\*Q

ans =

```
1.0000      0      -0.0000      -0.0000
      0      1.0000      0.0000      0.0000
-0.0000      0.0000      1.0000      0.0000
-0.0000      0.0000      0.0000      1.0000
```

>>Q\*R

ans =

```
-10.0000  13.0000   7.0000 -11.0000
 2.0000   1.0000  -5.0000  3.0000
-6.0000   3.0000  13.0000 -3.0000
16.0000 -16.0000  -2.0000  5.0000
 2.0000   1.0000  -5.0000 -7.0000
```